Bachelor project: Finite hyperfields

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1 Preliminaries

Hypergroups, hyperrings, and hyperfields are a generalization of ordinary groups, rings and fields, in which addition is allowed to be *multivalued*. To get an idea how such a notion may be used, think of the three possible signs a real number could have: $-1, 0, 1$. Suppose we want to define the addition '+' to make an additive group $S = (\{-1, 0, 1\}, 1, +)$ which expresses how signs are added. Adding two positive numbers always gives a positive number, so we would like that $1 + 1 = 1$. Also, adding negative numbers always gives a negative number, so $(-1) + (-1) = -1$. But adding a positive and a negative number may give positive, negative, or zero, and so there is no satisfactory way to use normal addition to express the addition of signs. If we would choose $1 + (-1) = 0$, then $S = (\{-1, 0, 1\}, 1, +)$ becomes a known additive group, $\mathbb{Z}_3$, but this is not quite the object we wanted.

The solution is to allow the sum of two elements to have more than one value. Formally, $x + y$ will then be a set, the set of possible outcomes of x plus y. What follows next is one way to formally generalize groups, rings, and fields along these lines. To make it very clear that the addition is unusual, we write $x \boxplus y$ rather than $x + y$.

A *hyperoperation* on G is a map $\boxplus : G \times G \to 2^G \setminus \{\emptyset\}$. Any hyperoperation induces a map $\overline{\boxplus} : 2^G \times 2^G \to 2^G$ by setting

$$X \overline{\boxplus} Y := \bigcup \{x \boxplus y : x \in X, y \in Y\}.$$

Slightly abusing notation, one writes $x \boxplus Y := \{x\} \overline{\boxplus} Y$, $X \boxplus y := X \overline{\boxplus} \{y\}$, and $X \boxplus Y := X \overline{\boxplus} Y$. The hyperoperation $\boxplus$ then is *associative* if $x \boxplus (y \boxplus z) = (x \boxplus y) \boxplus z$ for all $x, y, z \in G$. [1]

A *hypergroup* is a triple $(G, \boxplus, 0)$, where $0 \in G$ and $\boxplus : G \times G \to 2^G \setminus \{\emptyset\}$ is a commutative and associative hyperoperation, such that

(H0) $x \boxplus 0 = \{x\}$

(H1) for each $x \in G$ there is a unique $y \in G$ so that $0 \in x \boxplus y$; we denote this unique element by $-x := y$

(H2) $x \in y \boxplus z$ if and only if $z \in x \boxplus (-y)$

If G, H are hypergroups, then a map $f : G \to H$ is a *hypergroup homomorphism* if $f(x \boxplus y) \subseteq f(x) \boxplus f(y)$ for all $x, y \in G$, and $f(0) = 0$.

A *hyperring* is a tuple $(R, \cdot, \boxplus, 1, 0)$ so that

(R0) $(R, \boxplus, 0)$ is a hypergroup

(R1) $(R \setminus \{0\}, \cdot, 1)$ is monoid

(R2) $0 \cdot x = x \cdot 0 = 0$ for all $x \in R$

(R3) $\alpha(x \boxplus y) = \alpha x \boxplus \alpha y$ and $(x \boxplus y)\alpha = x\alpha \boxplus y\alpha$ for all $\alpha, x, y \in R$

[1] Note that if we do not extend $\boxplus$ in this way, the expression $x \boxplus (y \boxplus z)$ is not defined, as $(y \boxplus z)$ is a set!

If R, S are hyperrings, then $f : R \to S$ is a *hyperring homomorphism* if f is a hypergroup homomorphism, $f(1) = 1$, and $f(x \cdot y) = f(x) \cdot f(y)$ for all $x, y \in R$.

A *(skew) hyperfield* is a hyperring such that $0 \neq 1$, and each nonzero element has a multiplicative inverse. A hyperfield homomorphism is just a homomorphism of the underlying hyperrings.

The smallest hyperfield is the *Krasner hyperfield* $\mathbb{K} := (\{0,1\}, \cdot, \boxplus, 1, 0)$, with $1 \boxplus 1 = \{0, 1\}$.

The second-smallest is the *hyperfield of signs* (described above), $\mathbb{S} := (\{-1,0,1\}, \cdot, \boxplus, 1, 0)$, where $1 \boxplus 1 = \{1\}$, $-1 \boxplus -1 = \{-1\}$, and $1 \boxplus -1 = \{-1, 0, 1\}$.

The smallest non-abelian group can be fitted with a hyperaddition to form a skew hyperfield. Consider $\mathbb{D}_3 := (D_3 \cup \{0\}, \cdot, \boxplus, 1, 0)$, where $(D_3, \cdot, 1)$ is the dihedral group presented as $D_3 = \{d_i : i \in \mathbb{Z}_6\}$ with $1 := d_0$, with multiplication and hyperaddition fixed by

$$d_i \cdot d_j = \begin{cases} d_{i+j} & \text{if } i \in \{0,2,4\} \\ d_{i-j} & \text{if } i \in \{1,3,5\} \end{cases} \quad \text{and } d_i \boxplus d_j = \begin{cases} \{d_i\} & \text{if } j = i \\ \{d_i, d_j\} & \text{if } j = i+1 \\ \{d_i, d_{i+1}, d_j\} & \text{if } j = i+2 \\ D_3 \cup \{0\} & \text{if } j = i+3 \end{cases}$$

2 Goal of this bachelor project

Investigate finite skew hyperfields and their homomorphisms.

1. For any finite multiplicative group $(G, \cdot, 1)$, investigate if $G \cup \{0\}$ can be fitted with a hyperaddition $\boxplus$ so that $(G \cup \{0\}, 1, 0, \cdot, \boxplus)$ is hyperfield. If so, describe the possible hyperfields, and if not, show that it is impossible. Use pen and paper, or a computer, or both, whichever you prefer.

2. Investigate if operations on groups, such as dividing out a subgroup, induce similar operations on hyperfields.

3. Find operations to extend a hyperfield to a bigger one, or to recombine to hyperfields into another hyperfield. Describe the conditions under which your operations work.

4. Find constructions of infinite families of finite hyperfields.

3 End terms

To complete the project, you must study the literature on finite groups, so that you can enumerate and describe the smallest finite groups in order of increasing size. Then, you must at least execute (1) for the smallest groups. Finally, you must report on your results in writing, giving accurate descriptions of the hyperfields you found, proofs that these are indeed hyperfields, and proofs of any other claims you make. The project is finished if a presentation of about 20-25 minutes has been given. For further rules and conditions, see the Bachelor Project study guide on Canvas (2WH40).

4 Timing

The project must be executed in one or two quartiles. At the end of the agreed time frame, a grade will be given and the project will finish. There are three phases to any project, each taking roughly equal time:

1. study of the literature

2. research

3. writing a report, preparing a presentation

Before the start of the project we will make appointments for three meetings at the end of each phase.